M.Sc. (Mathematics) (New CBCS Pattern) Semester-II

# PSCMTH06 - Field Theory

P. Pages : 2 **GUG/S/25/13746**

Time : Three Hours * 2 2 2 1 * Max. Marks : 100

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Notes : 1. Solve all **five** questions.
2. Each question carry equal marks.

**UNIT – I**

**1.** a) If $f(x), g(x) \in R[x]$, where R is a UFD, then prove that $c(fg) = c(f)c(g)$. **10**

b) Prove that every PID is a UFD, but a UFD is not necessarily a PID. **10**

**OR**

c) Prove that an irreducible element in a commutative principal ideal domain is always prime. **10**

d) Prove that 3 is irreducible but not prime in the ring $\mathbb{Z}\left[\sqrt{-5}\right]$. **10**

**UNIT – II**

**2.** a) Let E be an algebraic extension of F, and let $\sigma : E \to E$ be an embedding of E into itself over F. Then prove that $\sigma$ is onto. **10**

b) Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x], n \geq 1$. If there is a prime p such that $p^2 \nmid a_0$, $p \mid a_0$, $p \mid a_1, \cdots\cdots, p \mid a_{n-1}, p \nmid a_n$, then prove that f(x) is irreducible over $\mathbb{Q}$. **10**

**OR**

c) Let $F \subseteq E \subseteq K$ be fields. If $[K:E] < \infty$ and $[E:F] < \infty$, then prove that **10**

i) $[K:F] < \infty$

ii) $[K:F] = [K:E]\,[E:F]$

d) Determine the minimal polynomials over $\mathbb{Q}$ of the following numbers. **10**

i) $\sqrt{2} + 5$

ii) $3\sqrt{2} + 5$

**UNIT – III**

**3.** a) If the multiplicative group F* of nonzero elements of a field F is cyclic, then prove that F is finite. **10**

b) Let K be a splitting field of the polynomial $f(x) \in F[x]$ over a field F. If E is another splitting field of f(x) over F, then prove that there exists an isomorphism $\sigma : E \to K$ that is Identity on F. **10**

**OR**

c) If E is a finite separable extension of a field F, then prove that E is a simple extension of F. **10**

d) Let E be an extension of a field F, and let $\alpha \in E$ be algebraic over F. Then prove that $\alpha$ is separable over F if and only if $F(\alpha)$ is a separable extension of F. **10**

## UNIT – IV

**4.** a) If $f(x) \in F[x]$ has r district roots in its splitting field E over F, then prove that the Galois group $G(E/F)$ of $f(x)$ is a subgroup of the symmetric group $S_r$. **10**

b) Prove that every polynomial $f(x) \in \mathbb{C}[x]$ factors into linear factor in $\mathbb{C}[x]$. **10**

**OR**

c) Prove that the Galois group of $x^4 - 2 \in \mathbb{Q}[x]$ is the octic group (= group of symmetries of square) **10**

d) Prove that the Galois group of $x^4 + 1 \in \mathbb{Q}[x]$ is the Klein four-group. **10**

**5.** Solve all the four questions.

a) Define **5**
   i) Prime element
   ii) Irreducible element.

b) Show that $x^2 - 2$ is irreducible over $\mathbb{Q}$. **5**

c) Define **5**
   i) Separable extension
   ii) Simple extension.

d) Define: **5**
   i) Galois Group
   ii) Galois extension.

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